COOPER'S THEOREMS

Early in my career I stopped doing mathematical research in the traditional sense and concentrated on developing mathematics curriculum. Being at a new university, as Macquarie University was in the 1970's I had enormous freedom to teach what I wanted, especially in the senior years.

As a result I started writing my notes and, in the course of so doing, I developed a certain amount of new material. It could hardly be called 'research' but these are some theorems that may be of interest that you probably won't find anywhere else.

THE CUBIC FIT METHOD (*Elementary Calculus*)

This is an improvement on Simpson's Rule that, provided you know the derivative of the function, gives more accurate results for a similar amount of work. It comes from fitting a cubic to a strip based on the ordinates and derivatives at the endpoints. It doesn't need an even number of strips and consists of the Simpson's Rule formula, plus a 'correction':

$$\int_a^b y\,dx \approx \text{SIMPSON'S ESTIMATE} - \frac{h^2}{12}\left[y'\right]_a^b$$

where h = width of the strips.

For example the percentage errors in the estimates of $\int_1^5 \sqrt{x}\, dx$ with 4 strips are:

Trapezium Rule	0.3%
Simpson's Rule	0.02%
Cubic Fit Method	0.004%

SIMPSON'S RULE FOR DOUBLE INTEGRALS (*Techniques of Calculus*)

To obtain an estimate of the double integral

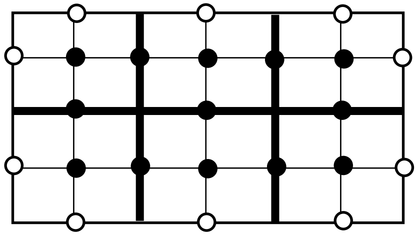
$$\int_c^d \left(\int_a^b \mathrm{f}(x, y)\, dx \right) dy$$

divide the rectangle $[a,b] \times [c,d]$ into smaller rectangles. There must be an even number in each direction so that each of these smaller rectangle is one quarter of a larger rectangle, which I call a **cell**.

The estimate is obtained by approximating the surface on each cell by a quadric surface, that is, by a surface with equation $z = ax^2 + by^2 + cxy + dx + ey + f$.

For each cell take the ordinates in the middle of the cell and in the middle of each side. An ordinate is a **boundary ordinate** if it is on the boundary of the entire rectangle. All others are called **internal ordinates**.

For example, if we take 3 cells horizontally and 2 vertically, the ordinates at the points marked with an white circle are boundary ordinates and those at the points marked with a black circle are internal ones.

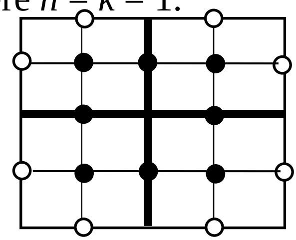


Theorem: An estimate of the integral of f(x, y) over the rectangle [a, b] × [c, d], where we divide this into $h \times k$ rectangle (h, k both being even) is:

$$\frac{2hk}{3}\left(2\Sigma \text{ internal ordinates} + \Sigma \text{ boundary ordinates}\right)$$

Example: Estimate the integral of x^3y over the rectangle [0, 4] × [0, 4] using 2 cells horizontally and 2 vertically.

Solution: Here $h = k = 1$.



Omitting the rows where the ordinate is zero:

x	*y*	**boundary**	**internal**
1	1		1
2	1		8
3	1		27
4	1	64	
1	2		2
3	2		54
1	3		3
2	3		24
3	3		81
4	3	192	
1	4	4	
3	4	108	
TOTALS		**368**	**200**

The estimate = $\frac{2}{3}(368 + 400) = 512$ which, surprisingly, is the exact value.
This is the exact value.

SECOND ORDER EXPANSION (*Matrices*)

This is an improvement on the cofactor method for evaluating determinants. While it involves somewhat less computation its main purpose is to simplify the proofs of the properties of determinants.

If A is a square matrix then $\delta_{ij}^{st}(A)$ is the matrix obtained from A by deleting rows *s*, *t* and columns *i*, *j*.

$$|A| = \sum_{i<j} (-1)^{1+i+j} \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} . |\delta_{ij}^{12}(A)| .$$

THE GENERALISED TRACE METHOD (*Matrices*)

This computes the coefficients of the characteristic polynomial of a square matrix, without having to evaluate $|\lambda I - A|$, with its error prone calculations, manipulating expressions in λ.

The ***k*-th trace**, **$tr_k(A)$**, is the sum of all the $k \times k$ sub-determinants that can be obtained from A by deleting corresponding rows and columns. So $tr_0(A) = 1$, $tr_1(A)$is the normal trace, and $tr_n(A)$, for an $n \times n$ matrix is just |A|.

The characteristic polynomial of the $n \times n$ matrix A is:

$$\chi(\lambda) = \sum_{0}^{n} (-1)^k \, tr_k(A) \lambda^{n-k} .$$

If $A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$, $tr_1(A) = \text{trace} = 15$,

$tr_2(A) = 3 + 12 + 3 = 18$ and $tr_3(A) = |A| = 0$.

So $\chi_A(\lambda) = \lambda^3 - 15\lambda^2 + 18\lambda$.

THE ONE-WAY EUCLIDEAN ALGORITHM (*Techniques of Algebra*)

The Euclidean Algorithm finds the GCD of two integers, *a* and *b*. However if you want to express this in the form *ah* + *bk* you have to work backwards through all these calculations. Here's a method whereby enough information is collected along the way to find a suitable *h* and *k* once the GCD is obtained.

Suppose we want to find GCD(92,24) and wish to express it as 92*h* + 24*k*.

a	*q*	*b*
217		**0**
91	2 = INT(217,91)	**1**
35 = 217 – 2.91	2 = INT(91,35)	–2 = 0 – 2.1
21 = 91 – 2.35	1 = INT(35,21)	5 = 1 – 2.(–2)
14 = 35 – 1.21	1 = INT(21,14)	–7 = –2 – 1.5
7 = 21 – 14.1	2 = INT(14,7)	12 = 5 – 1.(–7)
0 = 14 – 2.7		

The *a* column contain the successive remainders.
The pattern for the *b* column is similar:

	b_0
q	b_1
	$b_0 - qb_1$

The GCD is the last non-zero entry in the *a* column = 7.

k is the last entry in the b column = 12.
So $7 = 217.h + 91.12$. We can easily find $h = -5$.
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INTEGRATION CLOSED CLASSES OF FUNCTIONS (*Techniques of Calculus*)

$$\{a(x) + b(x)\sin x + c(x)\cos x\} \text{ and}$$

$$\left\{a(x) + b(x)\tan^{-1}x + c(x)\log(1 + x^2) + \frac{d(x)}{1 + x^2}\right\},$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are real polynomials, is closed under integration.

NEWTON'S METHOD FOR FUNCTIONS OF 2 VARIABLES (*Techniques of Calculus*)

I'd be surprised if this hasn't been done before but I can't find it, so I'm tentatively making a claim on it until I discover that indeed it's been done before.

If we wish to solve the system $\left.\begin{matrix} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{matrix}\right\}$ we begin with a guess (x_0, y_0) and calculate (x_1, y_1) as follows: $\left.\begin{matrix} x_1 = x_0 - e_x \\ y_1 = y_0 - e_y \end{matrix}\right\}$ where $\Delta e_x = \begin{vmatrix} z_1 & d_{1y} \\ z_2 & d_{2y} \end{vmatrix}$ and $\Delta e_y = \begin{vmatrix} d_{1x} & z_1 \\ d_{2x} & z_2 \end{vmatrix}$

where $\Delta = \begin{vmatrix} d_{1x} & d_{1y} \\ d_{2x} & d_{2y} \end{vmatrix}$.

Here $d_{1x} = \left(\frac{\partial f_1}{\partial x}\right)_0$, $d_{1y} = \left(\frac{\partial f_1}{\partial y}\right)_0$, $d_{2x} = \left(\frac{\partial f_2}{\partial x}\right)_0$, $d_{2y} = \left(\frac{\partial f_2}{\partial y}\right)_0$, the respective partial derivatives evaluated at (x_0, y_0), and $z_1 = f_1(x_0, y_0)$ and $z_2 = f_2(x_0, y_0)$. As with the one-variable Newton's Method we iterate, making (x_1, y_1) the new (x_0, y_0). Provided we're reasonably close to a solution this converges to one.

We can use this method to find complex solutions to equations $f(z) = 0$ by considering real and imaginary parts.

THE TOO MANY PRIMES TEST (Galois Theory)

There are many tests for primeness in an integer polynomial – none of them works in the majority of cases. The too many primes test is a useful addition to Eisenstein's method, and all the others.

If $f(x) \in \mathbb{Z}[x]$ has degree $n > 5$ and $f(m)$ is prime or ± 1 for at least $n + 3$ integer values of m, the $f(x)$ is prime over $\mathbb{Q}$. For $n = 4$ or 5 this target is 9. For $n = 2$ or 3, it is $n + 3$. (These targets are best possible.)

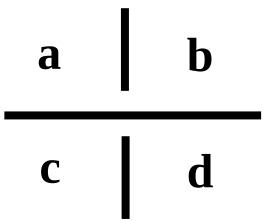
COLLINEARITY LEMMA (Geometry)

This is a useful lemma for simplifying certain proofs in Projective Geometry, including Desargue's Theorem and Pappus' Theorem. It is based on the real projective plane being thought of a 1- and 2-dimensional subspaces of R^3 (points and lines respectively).

If $P = \langle \mathbf{p} \rangle$, Q, R, S are collinear projective points such that P, Q, R are distinct and $P \neq S$, then for a suitably chosen vector $\mathbf{q}$ and scalar λ we may express the four points as: $P = \langle \mathbf{p} \rangle$, $Q = \langle \mathbf{q} \rangle$, $R = \langle \mathbf{p} + \mathbf{q} \rangle$, $S = \langle \lambda\mathbf{p} + \mathbf{q} \rangle$. Moreover ,if the Euclidean plane is embedded in $\mathbb{R}^3$ and P*, Q*, R* and S* are the corresponding points on the plane, λ is their cross ratio.

ALEXANDER GROUPS (Topology)

Let K be a knot and let M be a map for it. We define an abelian group for the knot in terms of generators and relations as follows. Assign a generator to each face, except for the outside and one adjacent face, which are both assigned 0. At each crossing create a relation as follows:



$\mathbf{a} + \mathbf{b} = \mathbf{c} + \mathbf{d}$

produces the relation $\mathbf{a + b = c + d}$. The abelian group with these generators and relations, which I call the **Alexander Group**, is an invariant for the knot. It can be generalized to links, and for knots it can be generalized to a module over the ring of rational Laurent polynomials in t, by using the relation:

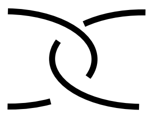
$$t(\mathbf{a} + \mathbf{b}) + \mathbf{c} + \mathbf{d} = \mathbf{0}$$

if the overcrossing is coming from the right. From this it is possible to obtain the Alexander polynomial.

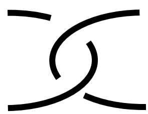
ALEXANDER GROUPS OF CHAINS (*Topology*)

Although this was proved by Simon Byrne, a vacation scholar that I supervised at Macquarie University, I played a small part in it.

A **chain** is a set of two or more non-intersecting closed curves in R^3 where each is linked to the next. (For example, the Olympic logo.) There are two ways that a pair of adjacent links can occur in a projection of a chain and we'll refer to these as positive and negative connections as follows:



+ve connection



–ve connection

This distinction only occurs at the level of projections because both are equivalent for the chain itself.

Suppose a chain with $n \geq 2$ links has its ends joined. In a projection onto a plane (where the only crossings are those that join each link to its neighbours) let m be the absolute difference between the number of positive and negative connections. The Alexander group of this closed chain is $\mathbb{Z}_2^{\,n-2} \oplus \mathbb{Z}_{2m}$ if $m > 0$ and

$$\mathbb{Z}_2^{\,n-2} \oplus \mathbb{Z} \text{ if } m = 0.$$

(Here $\mathbb{Z}_2^{\,n-2}$ denotes the direct sum of $n - 2$ copies of $\mathbb{Z}_2$.)

CLASS EQUATIONS (Group Theory)

Let $t{*}n$ denote n conjugacy classes of size n.

• Let G be a group of order 2N with a conjugacy class of size N. Then N is odd and the class equation for G is

$$2\mathrm{N} = 1 + 2{*}\left(\frac{\mathrm{N}-1}{2}\right) + \mathrm{N}.$$

• Let G be a group of order 3N with two conjugacy classes of size N. Then $|\mathrm{G}'| = \mathrm{N}$ and the class equation for G is $3\mathrm{N} = 1 + 3t_1 + 3t_2 + \ldots + 3t_k + \mathrm{N} + \mathrm{N}$ where the class equation for G′ is:

$$\mathrm{N} = 1 + t_1{*}3 + t_2{*}3 + \ldots + t_k{*}3.$$

• Let G be a group of order pN, where p is prime, with p–1 conjugacy classes of size N. Then G is a Frobenius group with kernel G′ of order N.

POWER AUTOMORPHISMS (*Group Theory*)

A power automorphism, θ, of a group is one that fixes every subgroup (i.e. $\theta(x) = x^n$ for all x, but the n may vary). Every power automorphism is central (induces the identity automorphism on G/Z(G)).

SYLOW SUBGROUPS OF SYMMETRIC GROUPS (*Group Theory*)

Let $G^{(r)}$ denote the wreath product of r copies of G and G^r the direct product of r copies of G. If p is prime and $N = a_r a_{r-1} \ldots a_1 \; a_0$ in base p notation, the Sylow p-subgroups of S_N are isomorphic to:

$C_p^{(r)a_r} \times C_p^{(r-1)a_{r-1}} \times \ldots \times C_p^{(2)a_2} \times C_p^{a_1}$, where C_p is the cyclic group of order p.

***p*-ORDER and *p*-INERTIA (*Number Theory*)**

If p is prime and is coprime with m, the ***p*-order** of m is **u(*p, m*)**. the smalles positive u such that $p^u \equiv 1 (\mathrm{mod}\ p)$. If p, q are distinct primes the ***p*-inertia** of q, denoted by **v(*p, q*)**, is the largest v such that

$p^{u(p,q)} \equiv 1 (\mathrm{mod}\ q^v)$.

If $2 < p < q$ are primes then:

$$u(p, q^t) = \begin{cases} u(p, q) \text{ if } 0 < t \le v(p, q) \\ u(p, q)\, q^{t - v(p,q)} \text{ if } t > v(p, q) \end{cases}.$$

THE UNIQUENESS OF 10 (*Number Theory*)

I thought I had a proof of this theorem, but mislaid it. The proof I use in my Number Theory Notes is due to a colleague, Gerry Myerson.

The number 10 is the only composite number such that all its positive divisors have the form $n^k + 1$ for $k > 1$.

THE GROUP OF NON-ZERO MULTIPLICATIVE FUNCTIONS (*Number Theory*)

A function F(n) on ℕ is multiplicative if F(mn) = F(m)F(n) whenever m, n are coprime. The **Möbius product** F ∗ G of two multiplicative functions F, G is defined by:

$$(\mathrm{F} * \mathrm{G})(\mathrm{n}) = \sum_{d|n} \mathrm{F}(d)\mathrm{G}\left(\frac{n}{d}\right) = \sum_{d|n} \mathrm{F}\left(\frac{n}{d}\right)\mathrm{G}(d) .$$

We can write this symmetrically as (F ∗ G)(n) = $\sum_{cd=n} \mathrm{F}(c)\mathrm{G}(d)$, and so F ∗ G = G ∗ F for all multiplicative functions.

The set of all non-zero multiplicative functions is an abelian group under the Möbius product. It has an identity $1:\mathbb{N} \to \mathbb{N}$ defined by $1(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ otherwise} \end{cases}$.